Integration by Parts

Liming Pang

Integration by Parts is a useful technique in evaluating integrals, which is based on the Leibniz Rule of Differentiation.

Theorem 1. *(Integration by Parts)*

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

Proof. By the Leibniz Rule of differentiating a product of functions, we know

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

So $f(x)g(x)$ is an antiderivative of $f'(x)g(x) + f(x)g'(x)$,

$$\int f'(x)g(x) + f(x)g'(x)\,dx = f(x)g(x) + C$$

We then see

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

□

Remark 2. Recall that given a differentiable function f, there is a corresponding differential $df = f'(x)dx$, so the aove theorem can also be written as

$$\int f(x)\,dg(x) = f(x)g(x) - \int g(x)\,df(x)$$

We can use the above theorem to find antiderivatives of product of functions.

Example 3. *Find antiderivatives of $f(x) = xe^x$.*

$$\begin{aligned}\int xe^x\,dx &= \int x(e^x)'\,dx \\ &= xe^x - \int (x)'e^x\,dx \\ &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C\end{aligned}$$

If you like to use the differential notation instead, you will get

$$\begin{aligned}\int xe^x\,dx &= \int x\,de^x \\ &= xe^x - \int e^x\,dx \\ &= xe^x - e^x + C\end{aligned}$$

Exercise 4. *Find antiderivatives of $f(x) = x^2e^x$.*

Example 5. *Compute $\int \frac{1}{x}\ln x\,dx$*

$$\begin{aligned}\int \frac{1}{x}\ln x\,dx &= \int (\ln x)'\ln x\,dx \\ &= (\ln x)(\ln x) - \int \ln x(\ln x)'\,dx \\ &= (\ln x)^2 - \int \frac{1}{x}\ln x\,dx\end{aligned}$$

We see $\int \frac{1}{x}\ln x\,dx = \frac{1}{2}(\ln x)^2 + C$

Exercise 6. *Can you compute the above integral $\int \frac{1}{x}\ln x\,dx$ using Integration by Substitution?*

Example 7. *Evaluate* $\int e^x \cos x \, dx$

$$\begin{aligned}
\int e^x \cos x \, dx &= \int e^x \, d \sin x \\
&= e^x \sin x - \int \sin x \, de^x \\
&= e^x \sin x - \int e^x \sin x \, dx \\
&= e^x \sin x - (-\int e^x \, d \cos x) \\
&= e^x \sin x + \int e^x \, d \cos x \\
&= e^x \sin x + e^x \cos x - \int \cos x \, de^x \\
&= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx
\end{aligned}$$

So we obtain $2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$, *and finally divide by* 2 *and add constant, we get* $\int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x)$

Exercise 8. *Evaluate* $\int e^x \sin x \, dx$

Example 9. *Evaluate* $\int \ln x \, dx$

$$\int \ln x \, dx = x \ln x - \int x \, d \ln x = x \ln x - \int x (\ln x)' \, dx = x \ln x - \int 1 \, dx = x \ln x - x + C$$

We can also apply the method of Integration by Parts in evaluating definite integrals:

Theorem 10.

$$\int_a^b f(x) g'(x) \, dx = f(b)g(b) - f(a)g(a) - \int_a^b g(x) f'(x) \, dx$$

Proof. We have seen that $f(x)g(x)$ is an antiderivative of $f(x)g'(x) + f'(x)g(x)$. Then by the Fundamental Theorem of Calculus,

$$\int_a^b f(x)g'(x) + f'(x)g(x) \, dx = f(b)g(b) - f(a)g(a)$$

□

Example 11. *Evaluate* $\int_1^3 x \ln x \, dx$

$$\begin{aligned}
\int_1^3 x \ln x \, dx &= \int_1^3 \ln x d\frac{x^2}{2} \\
&= \frac{x^2}{2} \ln x \Big|_1^3 - \int_1^3 \frac{x^2}{2} \, d \ln x \\
&= \frac{9}{2} \ln 3 - \int_1^3 \frac{x^2}{2} \frac{1}{x} \, dx \\
&= \frac{9}{2} \ln 3 - \int_1^3 \frac{x}{2} \, dx \\
&= \frac{9}{2} \ln 3 - \frac{x^2}{4} \Big|_1^3 \\
&= \frac{9}{2} \ln 3 - 2
\end{aligned}$$

Example 12. *Evaluate* $\int_0^1 \tan^{-1} x \, dx$

$$\begin{aligned}
\int_0^1 \tan^{-1} x \, dx &= x \tan^{-1} x \Big|_0^1 - \int_0^1 x \, d \tan^{-1} x \\
&= \tan^{-1} 1 - \int_0^1 \frac{x}{1 + x^2} \, dx \\
&= \frac{\pi}{4} - \frac{1}{2} \int_0^1 \frac{1}{1 + x^2} \, d(1 + x^2) \\
&= \frac{\pi}{4} - \frac{1}{2} \ln(1 + x^2) \Big|_0^1 \\
&= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - \ln 1) \\
&= \frac{\pi}{4} - \frac{\ln 2}{2}
\end{aligned}$$